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Miura map and Bi-Hamiltonian formulation for restricted flows of the KdV hierarchy

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Abstract. The first Hamiltonian structure and the Miura map are constructed for the restricted flows of Korteweg–de Vries (KdV) hierarchy and its modification. On this basis the second Hamiltonian structure of KdV restricted flows is derived.

1. Introduction

The bi-Hamiltonian formulation of dynamics seems nowadays to be one of the most important features of integrability. First recognized and applied to infinite-dimensional systems (nonlinear partial differential equations (NPDE) in $1+1$, i.e. so called soliton systems), it has recently been developed in the theory of finite systems of classical mechanics. One of the efficient ways of constructing new integrable bi-Hamiltonian finite systems is a restriction of infinite-dimensional integrable systems to finite dimensional invariant submanifolds. A few special cases of such restrictions have been considered up to now. First, one has the so-called stationary flows of soliton systems, for which integrability [1] and bi-Hamiltonian formulations [2] have been proved. Second, are finite-dimensional integrable systems obtained by the nonlinearization of the Lax equation of soliton system under certain constraints between potentials and eigenfunctions [3–6]. For these systems the bi-Hamiltonian formulation has been found [7–9]. In this paper we consider the more general finite-dimensional restrictions of NPDE [8] which contains the previous ones as special cases. Moreover, we present a systematic way of constructing the bi-Hamiltonian structure of such systems by applying here the finite dimensional version of the so-called Miura maps.

In section 2 we derive the first Hamiltonian formulation of the KdV and the MKdV restricted flows dynamics. In section 3 we present the appropriate Miura map relating both restricted flows dynamics. Then, following the standard procedure, we apply the Miura map to the first Hamiltonian structure of the modified system and generate the second Hamiltonian structure of a given system. Section 4 consists of examples of the first few restricted flows of the KdV hierarchy and of its bi-Hamiltonian representation. In the conclusions we give few arguments which suggest that the method is more general and could be applied to other systems as well.

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2. First Hamiltonian structure of KdV restricted flows and its modifications

For the KdV hierarchy of equations

$$u_{t_m} = K_m[u] \qquad m = 0, 1, 2, \ldots \tag{2.1}$$

which follows from the spectral problem

$$(\partial_x^2 + u)\Psi = \lambda\Psi \tag{2.2}$$

restricted flows were defined [7] as the set of equations

$$\Psi_{kxx} + u\Psi_k = \lambda_k\Psi_k \qquad k = 1, \ldots, N \tag{2.3}$$

$$\alpha B_0\left(\sum_{i=1}^{N} \Psi_i^2\right) = K_m[u] \qquad \alpha = \text{const} \tag{2.4}$$

where (2.3) are N copies of the spectral problem (2.2), $B_0 = \partial_x$ is the first Hamiltonian operator of KdV hierarchy (2.1) and (2.4) is the restriction of a general KdV symmetry $B_0(\delta\lambda/\delta u)$, related to the hierarchy (2.1) through the asymptotic ($\lambda \to \infty$) expansion

$$B_0\frac{\delta\lambda}{\delta u} = (\Psi^2)_x = \sum_{n=0}^{\infty} K_n\lambda^{-n} \tag{2.5}$$

to a particular $K_m[u]$.

Our first goal is to find a Hamiltonian formulation for the dynamics (2.3)-(2.4). To do this let us rewrite them in a more convenient form

$$\begin{pmatrix}\Psi_{1k}\\ \Psi_{2k}\end{pmatrix}_x = \begin{pmatrix}0 & 1\\ \lambda_k - u & 0\end{pmatrix}\begin{pmatrix}\Psi_{1k}\\ \Psi_{2k}\end{pmatrix} \qquad \Psi_{2k} = \Psi_{1kx} \qquad k = 1, \ldots, N \tag{2.6}$$

$$\gamma_m[u] = \left(\sum_{i=1}^{N} \Psi_i^2\right) + c \tag{2.7}$$

where $\gamma_m(u)$ are conserved one-forms of the KdV, such that $B_0\gamma_m[u] = K_m[u]$, c is a constant of integration and the factor $\alpha = 1$ is chosen for convenience. Before we treat this system let us consider for a moment an uncoupled version

$$\begin{pmatrix}\Psi_{1k}\\ \Psi_{2k}\end{pmatrix}_x = \begin{pmatrix}0 & 1\\ \lambda_k & 0\end{pmatrix}\begin{pmatrix}\Psi_{1k}\\ \Psi_{2k}\end{pmatrix} \qquad k = 1, \ldots, N \tag{2.8}$$

$$\gamma_m[u] = c \tag{2.9}$$

The first system is the asymptotic ($u \to 0$) KdV Lax problem and the second one is the nth stationary flow of the KdV. Both of them can be put into a Hamiltonian form. Let $\Psi_{1k} = Q_k$, $\Psi_{2k} = P_k$, $Q = (Q_1, \ldots, Q_N)^T$, $P = (P_1, \ldots, P_N)^T$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$, then

$$\begin{pmatrix}Q\\ P\end{pmatrix}_x = \begin{pmatrix}0 & I_N\\ -I_N & 0\end{pmatrix}\nabla(\tfrac{1}{2}P^TP - \tfrac{1}{2}Q^T\Lambda Q) \equiv \Pi\nabla\mathscr{H}_L \tag{2.10}$$

where I_N is the $N \times N$ unit matrix and ∇ stands for the gradient operator. On the other hand the stationary flow (2.9) can be put into a Hamiltonian form according to the procedure presented in [2]. Actually

$$\begin{pmatrix}q\\ p\\ c\end{pmatrix}_x = \begin{pmatrix}K_{qm}\\ K_{pm}\\ 0\end{pmatrix} = \begin{pmatrix}0 & I_m & 0\\ -I_m & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\nabla\mathscr{H}_{sm}(q, p, c) \tag{2.11}$$

where $q=(q_1,\ldots,q_m)^T$, $p=(p_1,\ldots,p_m)^T$, $q_k=u_{kx}$ and $p_k=p_k(u,u_x,\ldots)$ are conjugate variables.

Now, let us introduce the coupling between both systems through the interacting term in the Hamiltonian

$$\mathcal{H}_{\rm int}=\tfrac{1}{2}q_1Q^TQ. \tag{2.12}$$

This gives the following new Hamiltonian system

$$\begin{pmatrix} Q\\ q\\ P\\ p\\ c\end{pmatrix}_x=\begin{pmatrix} P\\ K_{qm}\\ (\Lambda-q_1I_N)Q\\ K_{pm}+K_{\rm int}\\ 0\end{pmatrix}=\begin{pmatrix} 0 & 0 & I_N & 0 & 0\\ 0 & 0 & 0 & I_N & 0\\ -I_N & 0 & 0 & 0 & 0\\ 0 & -I_N & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\nabla\mathcal{H}_m \tag{2.13}$$

where

$$K_{\rm int}=\begin{pmatrix} -\frac{1}{2}Q^TQ\\ 0\\ \vdots\\ 0\end{pmatrix}\qquad \mathcal{H}_m=\mathcal{H}_L+\mathcal{H}_{sm}+\mathcal{H}_{\rm int} \tag{2.14}$$

which is just the first Hamiltonian formulation of the restricted flow dynamics (2.6)-(2.7). It is interesting to notice that, in the case when $m=-1$, $K_{-1}[u]=0$ and our system (2.3)-(2.4) turns out to be the Neuman system [3], whereas in the case when $m=0$, $K_0[u]=u_x$ and after eliminating variable u we get the Garnier system [4] and in the case of $m=1$, $K_1[u]=\frac{1}{4}u_{xxx}+\frac{3}{2}uu_x$ and we deal exactly with the stationary flow of the Melnikov system [10].

The candidates for a modified system are of course restricted flows of the MKdV hierarchy

$$\begin{pmatrix}\varphi_{1k}\\ \varphi_{2k}\end{pmatrix}_x=\begin{pmatrix} v & \lambda_k\\ 1 & -v\end{pmatrix}\begin{pmatrix}\varphi_{1k}\\ \varphi_{2k}\end{pmatrix}\qquad k=1,\ldots,N \tag{2.15}$$

$$\bar{\alpha}\bar{B}_0\left(-\tfrac{1}{2}\sum_{i=1}^{N}\varphi_{1k}\varphi_{2k}\right)=\bar{K}_m[v]\qquad \bar{B}_0=-\partial_x \tag{2.16}$$

where (2.15) are N copies of the MKdV spectral problem and (2.16) is the restriction of a general MKdV symmetry $\bar{B}_0(\delta\lambda/\delta u)=\frac{1}{2}(\varphi_1\varphi_2)_x$ to a particular one $\bar{K}_m[v]$ for its hierarchy.

First, let us rewrite the system (2.15)-(2.16) in a more convenient form, integrating equation (2.16) once

$$\begin{pmatrix}\varphi_{1k}\\ \varphi_{2k}\end{pmatrix}_x=\begin{pmatrix} v & \lambda_k\\ 1 & -v\end{pmatrix}\begin{pmatrix}\varphi_{1k}\\ \varphi_{2k}\end{pmatrix}\qquad k=1,\ldots,N \tag{2.17}$$

$$\bar{\gamma}_m[v]=\bar{c}+\tfrac{1}{2}\sum_{i=1}^{N}\varphi_{1i}\varphi_{2i} \tag{2.18}$$

where $\bar{\gamma}_m[v]$ are conserved one-forms of the MKdV, $\bar{c}$ is a constant of integration and the factor $\bar{\alpha}=1$ is chosen for convenience.

Again, let us write down the uncoupled case

$$\begin{pmatrix}\varphi_{1k}\\ \varphi_{2k}\end{pmatrix}_x=\begin{pmatrix}0 & \lambda_k\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varphi_{1k}\\ \varphi_{2k}\end{pmatrix} \qquad k=1,\ldots,N \tag{2.19}$$

$$\bar{\gamma}_m[v]=\bar{c} \tag{2.20}$$

where (2.19) represents the asymptotic MKdV Lax problem and (2.20) the MKdV stationary flows. Let $\varphi_{1k}=\bar{Q}_k$, $\varphi_{2k}=\bar{P}_k$, $\bar{Q}=(\bar{Q}_1,\ldots,\bar{Q}_N)^T$, $\bar{P}=(\bar{P}_1,\ldots,\bar{P}_N)^T$; $\Lambda=\operatorname{diag}(\lambda_1,\ldots,\lambda_N)$, then the Hamiltonian form of (2.19) is the following

$$\begin{pmatrix}\bar{Q}\\ \bar{P}\end{pmatrix}_x=\begin{pmatrix}\Lambda\bar{P}\\ \bar{Q}\end{pmatrix}=\begin{pmatrix}0 & I_N\\ -I_N & 0\end{pmatrix}\nabla(\tfrac{1}{2}\bar{P}^T\Lambda\bar{P}-\tfrac{1}{2}\bar{Q}^T\bar{Q})\equiv\bar{\Pi}\nabla\bar{\mathcal{H}}_L. \tag{2.21}$$

We can do the same with the stationary flows. Applying the procedure from [2] we can transform (2.20) to the Hamiltonian form

$$\begin{pmatrix}\bar{q}\\ \bar{p}\\ \bar{c}\end{pmatrix}_x=\begin{pmatrix}\bar{K}_{\bar{q}m}\\ \bar{K}_{\bar{p}m}\\ 0\end{pmatrix}=\begin{pmatrix}0 & I_m & 0\\ -I_m & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\nabla\bar{\mathcal{H}}_{sm}(\bar{q},\bar{p},\bar{c}) \tag{2.22}$$

where $\bar{q}=(\bar{q}_1,\ldots,\bar{q}_m)^T$, $\bar{p}=(\bar{p}_1,\ldots,\bar{p}_N)^T$, $\bar{q}_k=v_{kx}$ and $\bar{p}_k=\bar{p}_k(v,v_x,\ldots)$ are conjugate variables.

Now, we shall again introduce the coupling between both systems through the interacting term in the Hamiltonian

$$\bar{\mathcal{H}}_{\text{int}}=\bar{q}_1\bar{Q}^T\bar{P} \tag{2.23}$$

which leads to the following Hamiltonian dynamics

$$\begin{pmatrix}\bar{Q}\\ \bar{q}\\ \bar{P}\\ \bar{p}\\ \bar{c}\end{pmatrix}_x=\begin{pmatrix}\Lambda\bar{P}+\bar{q}_1\bar{Q}\\ \bar{K}_{\bar{q}m}\\ \bar{Q}-\bar{q}_1\bar{P}\\ \bar{K}_{\bar{p}m}+\bar{K}_{\text{int}}\\ 0\end{pmatrix}=\begin{pmatrix}0 & 0 & I_N & 0 & 0\\ 0 & 0 & 0 & I_m & 0\\ -I_N & 0 & 0 & 0 & 0\\ 0 & -I_m & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\nabla\bar{\mathcal{H}}_m \tag{2.24}$$

where

$$\bar{\mathcal{H}}_m=\bar{\mathcal{H}}_L+\bar{\mathcal{H}}_{sm}+\bar{\mathcal{H}}_{\text{int}} \qquad \bar{K}_{\text{int}}=(-\bar{Q}^T\bar{P},0,\ldots,0)^T \tag{2.25}$$

and which is equivalent to our modified system (2.15)-(2.16).

3. Miura map and the second Hamiltonian structure of the KdV restricted flows

In the previous section we have derived, in a systematic way, the rst Hamiltonian formulation of the KdV and the MKdV restricted flows dynamics. Now, to derive the second Hamiltonian formulation, we have to find an explicit form of the Miura map relating the modified system (2.17) to a given one (2.10).

The Miura map between uncoupled systems has the well known form

$$M:\ Q=\bar{Q}$$

$$P=\Lambda\bar{P}$$

$$q=M_{qm}(\bar{q},\bar{p},\bar{c})$$

$$\begin{aligned} p &= M_{pm}(\bar{q}, \bar{p}, \bar{c}) \\ c &= M_{cm}(\bar{q}, \bar{p}, \bar{c}) \end{aligned} \tag{3.1}$$

where the first two equalities come from the gauge relation between the asymptotic of Lax equations and the other equalities are a consequence of the Miura map $u = -v^2 - v_x$ between the fields v and u.

Observation. The Miura map relating the modified coupled system (2.24) to a given one (2.13) is of the following form:

$$\begin{aligned} M\colon\ Q &= \bar{Q} \\ q &= M_{qm}(\bar{q}, \bar{p}, \bar{c}) \\ P &= \Lambda\bar{P} + \bar{q}_1\bar{Q} \\ p &= M_{pm}(\bar{q}, \bar{p}, \bar{c}) + M_{\text{int}} \\ c &= M_{cm}(\bar{q}, \bar{p}, \bar{c}) + \tfrac{1}{2}\bar{P}^T\Lambda\bar{P} - \tfrac{1}{2}\bar{Q}^T\bar{Q} + \bar{q}_1\bar{Q}^T\bar{P} \end{aligned} \tag{3.2}$$

where $M_{\text{int}} = (-\frac{1}{4}\bar{Q}^T\bar{P}, 0, \ldots, 0)^T$.

The proof is through direct calculation.

Now, by applying the map M of (3.2) to the first Hamiltonian structure of the MKdV restricted flows we generate the second Hamiltonian structure of the KdV restricted flows. For arbitrary N and m we find

$$\Pi_1 = M'\bar{\Pi}_0 M'^T$$

$$= \left(\begin{array}{c|c|c|c|c} 0 & 0 & \Lambda & -\frac{1}{4}Q \quad 0 & Q_x \\ \hline 0 & \left(\frac{\partial q}{\partial \bar{q}}\right)\left(\frac{\partial q}{\partial \bar{p}}\right)^T - \left(\frac{\partial q}{\partial \bar{p}}\right)\left(\frac{\partial q}{\partial \bar{q}}\right)^T & -\left(\frac{\partial q}{\partial p_1}\right)Q^T & \left(\frac{\partial q}{\partial \bar{q}}\right)\left(\frac{\partial p}{\partial \bar{p}}\right)^T - \left(\frac{\partial q}{\partial \bar{p}}\right)\left(\frac{\partial p}{\partial \bar{q}}\right)^T & q_x \\ \hline -\Lambda & -Q\left(\frac{\partial q}{\partial \bar{p}_1}\right)^T & 0 & \frac{1}{4}P \quad \frac{\partial p_2}{\partial \bar{p}_1}Q \quad \cdots \quad \frac{\partial p_m}{\partial \bar{p}_1}Q & P_x \\ \hline \begin{array}{c} \frac{1}{4}Q^T \\ 0 \end{array} & \left(\frac{\partial p}{\partial \bar{q}}\right)\left(\frac{\partial q}{\partial \bar{p}}\right)^T - \left(\frac{\partial p}{\partial \bar{p}}\right)\left(\frac{\partial q}{\partial \bar{q}}\right)^T & \begin{array}{c} -\frac{1}{4}P^T \\ \frac{\partial p_2}{\partial \bar{p}_1}Q^T \\ \vdots \\ \frac{\partial p_m}{\partial \bar{p}_1}Q^T \end{array} & \left(\frac{\partial p}{\partial \bar{q}}\right)\left(\frac{\partial p}{\partial \bar{p}}\right)^T - \left(\frac{\partial p}{\partial \bar{p}}\right)\left(\frac{\partial p}{\partial \bar{q}}\right)^T & p_x \\ \hline -Q_x^T & -q_x^T & -P_x^T & -p_x^T & 0 \end{array}\right) \tag{3.3}$$

Remark. As for other finite-dimensional restrictions of soliton hierarchies, the first Hamiltonian structure of the KdV and the MKdV restricted flows is canonical but degenerate, due to the introduction of an additional variable c, crucial in the construction of the second Hamiltonian structure. This again confirms the close relation between bi-Hamiltonian formulations of dynamics and the degeneracy of Hamiltonian (Poisson) structures for finite-dimensional systems.

4. Examples

4.1. The case m = 0

Let us consider the following system

$$Q_{xx} = (\Lambda - u)Q \tag{4.1}$$

$$u_x = (Q^T Q)_x \Rightarrow u = Q^T Q + c. \tag{4.2}$$

After eliminating the variable u we get the Garnier system [4] whose first Hamiltonian form is

$$\begin{aligned}\begin{pmatrix} Q \\ P \\ c \end{pmatrix}_x &= \begin{pmatrix} P \\ (\Lambda - c - Q^T Q)Q \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & I_N & 0 \\ -I_N & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \nabla\left(\tfrac{1}{2}P^T P + \tfrac{1}{4}(Q^T Q)^2 - \tfrac{1}{2}cQ^T Q - \tfrac{1}{2}Q^T \Lambda Q\right) \\ &\equiv \Pi_0 \nabla \mathcal{H}.\end{aligned} \tag{4.3}$$

In the modified case, we have $\bar{K}_0[v] = v_x$, hence

$$\begin{aligned}\bar{Q}_x &= \Lambda \bar{P} + v\bar{Q} \\ \bar{P}_x &= \bar{Q} - v\bar{P} \\ v_x &= \tfrac{1}{2}(\bar{Q}^T \bar{P})_x \Rightarrow v = \tfrac{1}{2}\bar{Q}^T \bar{P} + \bar{c}\end{aligned} \tag{4.4}$$

which, after elimination of the variable v, turns into the modified Garnier system

$$\begin{aligned}\begin{pmatrix} \bar{Q} \\ \bar{P} \\ \bar{c} \end{pmatrix}_x &= \begin{pmatrix} \Lambda \bar{P} - \tfrac{1}{2}(\bar{Q}^T \bar{P})\bar{Q} + \bar{c}\bar{Q} \\ \bar{Q} + \tfrac{1}{2}(\bar{Q}^T \bar{P})\bar{P} - \bar{c}\bar{P} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & I - N & 0 \\ -I_N & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \nabla\left(\tfrac{1}{2}\bar{P}^T \Lambda \bar{P} - \tfrac{1}{4}(\bar{Q}^T \bar{P})^2 + \bar{c}\bar{Q}^T \bar{P} - \tfrac{1}{2}\bar{Q}^T \bar{Q}\right) \\ &\equiv \bar{\Pi}_0 \nabla \bar{\mathcal{H}}.\end{aligned} \tag{4.5}$$

The appropriate Miura map is of the form

$$\begin{aligned} M:\ Q &= \bar{Q} \\ P &= \Lambda \bar{P} - \tfrac{1}{2}(\bar{Q}^T \bar{P})\bar{Q} + \bar{c}\bar{Q} \\ c &= \tfrac{1}{2}\bar{P}^T \Lambda \bar{P} - \tfrac{1}{4}(\bar{Q}^T \bar{P})^2 - \tfrac{1}{2}\bar{Q}^T \bar{Q} + \bar{c}\bar{Q}^T \bar{P} - \bar{c}^2\end{aligned} \tag{4.6}$$

recently found by Y Zeng [11], and leads directly to the second Hamiltonian structure of the Garnier system derived in [7] with the help of another method.

4.2. *The case of m = 1*

As the $K_1[u]=\frac{1}{4}u_{xxx}+\frac{3}{2}uu_x$, the restricted flows (2.6)-(2.7) can be put into the following Hamiltonian form

$$
\begin{pmatrix} Q \\ q \\ P \\ p \\ c \end{pmatrix}_x = \begin{pmatrix} P \\ -8p \\ (\Lambda - qI_N)Q \\ \frac{3}{8}q^2-\frac{1}{2}c-\frac{1}{2}Q^TQ \\ 0 \end{pmatrix}
= \left(\begin{array}{cc|cc|c} & & I_N & 0 & \\ \multicolumn{2}{c|}{0} & & & 0 \\ & & 0 & 1 & \\ \hline -I_N & 0 & & & \\ & & \multicolumn{2}{c|}{0} & 0 \\ 0 & -1 & & & \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 \end{array}\right) \nabla\left(\tfrac{1}{2}P^TP-\tfrac{1}{2}Q^T\Lambda Q+\tfrac{1}{2}qQ^TQ-\tfrac{1}{8}q^3+\tfrac{1}{2}cq-4p^2\right)
\equiv \Pi_0 \mathscr{H} \tag{4.7}
$$

where $q=u$, $p=-\frac{1}{8}u_x$, $c=\frac{1}{4}u_{xx}+\frac{3}{4}u^2-Q^TQ$.

Because $\bar{K}_1[v]=\frac{1}{4}v_{xxx}-\frac{3}{2}v^2v_x$, the modified restricted flows (2.17)-(2.18) have the Hamiltonian form

$$
\begin{pmatrix} \bar{Q} \\ \bar{q} \\ \bar{P} \\ \bar{p} \\ \bar{c} \end{pmatrix}_x = \begin{pmatrix} \Lambda\bar{P}+\bar{q}\bar{Q} \\ 2\bar{p} \\ \bar{Q}-\bar{q}\bar{P} \\ \bar{q}^3-2\bar{c}-\bar{Q}^T\bar{P} \\ 0 \end{pmatrix}
= \left(\begin{array}{cc|cc|c} & & I_N & 0 & \\ \multicolumn{2}{c|}{0} & & & 0 \\ & & 0 & 1 & \\ \hline -I_N & 0 & & & \\ & & \multicolumn{2}{c|}{0} & 0 \\ 0 & -1 & & & \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 \end{array}\right) \nabla\left(\tfrac{1}{2}\bar{P}^T\Lambda\bar{P}-\tfrac{1}{2}\bar{Q}^T\bar{Q}+\bar{q}\bar{Q}^T\bar{P}+\bar{p}^2-\tfrac{1}{4}\bar{q}^4+2\bar{c}\bar{q}\right)
=\bar{\Pi}_0\bar{\mathscr{H}} \tag{4.8}
$$

where $\bar{q}=v$, $\bar{p}=\frac{1}{2}v_x$, $\bar{c}=-\frac{1}{4}v_{xx}+\frac{1}{2}v^3-\frac{1}{2}\bar{Q}^T\bar{P}$.

The Miura map relating systems (4.8) to (4.7) reads

$$
\begin{aligned}
M:\ Q &= \bar{Q} \\
q &= -2\bar{p}-\bar{q}^2 \\
P &= \Lambda\bar{P}+\bar{q}\bar{Q}
\end{aligned}
$$

$$p=\tfrac{1}{4}\bar{q}^3+\tfrac{1}{2}\bar{q}\bar{p}-\tfrac{1}{2}\bar{c}-\tfrac{1}{4}\bar{Q}^T\bar{P}$$
$$c=\bar{p}^2-\tfrac{1}{4}\bar{q}^4+2\bar{q}\bar{c}+\tfrac{1}{2}\bar{P}^T\Lambda\bar{P}-\tfrac{1}{2}\bar{Q}^T\bar{Q}+\bar{q}\bar{Q}^T\bar{P}. \tag{4.9}$$

This leads to the second Hamiltonian structure of (4.7)

$$\Pi_1=M'\bar{\Pi}_0M'^T$$
$$=\begin{pmatrix} 0 & 0 & \Lambda & -\tfrac{1}{4}Q & P \\ 0 & 0 & -2Q^T & -\tfrac{1}{2}q & -8p \\ -\Lambda & 2Q & 0 & \tfrac{1}{4}P & (\Lambda-qI_N)Q \\ \tfrac{1}{4}Q^T & \tfrac{1}{2}q & -\tfrac{1}{4}P^T & 0 & \tfrac{3}{8}q^2-\tfrac{1}{2}c+\tfrac{1}{2}Q^TQ \\ -P^T & 8p & (qI_N-\Lambda)Q^T & -\tfrac{3}{8}q^2+\tfrac{1}{2}c-\tfrac{1}{2}Q^TQ & 0 \end{pmatrix} \tag{4.10}$$

derived by other method in [12].

4.3. *The case of* $m=2$

This case is new. We have

$$K_2[u]=(\gamma_2[u])_x=(\tfrac{1}{16}u_{4x}+\tfrac{5}{8}uu_{xx}+\tfrac{5}{16}u_x^2+\tfrac{5}{8}u^3)_x \tag{4.11}$$
$$\bar{K}_2[v]=(-\bar{\gamma}_2[v])_x=(\tfrac{1}{16}v_{4x}-\tfrac{5}{8}vv_x^2-\tfrac{5}{8}v^2v_{xx}+\tfrac{3}{8}v^5)_x \tag{4.12}$$

hence, the appropriate first Hamiltonian formulation of the KdV and the MKdV restricted flows are the following

$$\begin{pmatrix} Q \\ q_1 \\ q_2 \\ P \\ p_1 \\ p_2 \\ c \end{pmatrix}_x = \begin{pmatrix} P \\ q_2 \\ 32p_2 \\ (\Lambda-q_1I_N)Q \\ \tfrac{5}{16}q_1^3-\tfrac{5}{32}q_2^2-\tfrac{1}{2}c-\tfrac{1}{2}Q^TQ \\ -\tfrac{5}{16}q_1q_2-p_1 \\ 0 \end{pmatrix} \equiv \mathscr{K}$$

$$= \left(\begin{array}{cc|cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & I_N & 0 & \\ & & 0 & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & 0 \\ \hline -I_N & 0 & & & \\ 0 & \begin{matrix}-1 & 0\\ 0 & -1\end{matrix} & \multicolumn{2}{c|}{0} & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 \end{array}\right) \nabla(\tfrac{1}{2}P^TP-\tfrac{1}{2}Q^T\Lambda Q$$
$$+\tfrac{1}{2}q_1Q^TQ+16p_2^2+q_2p_1+\tfrac{5}{32}q_1q_2^2-\tfrac{5}{64}q_1^4+\tfrac{1}{2}cq_1)\equiv\Pi_0\mathscr{K} \tag{4.13}$$

where $q_1=u$, $q_2=u_x$, $p_1=-\tfrac{5}{16}uu_x-\tfrac{1}{32}u_{3x}$, $p_2=\tfrac{1}{32}u_{xx}$, $c=\tfrac{1}{16}u_{4x}+\tfrac{5}{8}uu_{xx}+\tfrac{5}{16}u_x^2+\tfrac{5}{8}u^3-Q^TQ$, and

$$\begin{pmatrix} \bar{Q} \\ \bar{q}_1 \\ \bar{q}_2 \\ \bar{P} \\ \bar{p}_1 \\ \bar{p}_2 \\ \bar{c} \end{pmatrix}_x = \begin{pmatrix} \Lambda\bar{P}+\bar{q}_1\bar{Q} \\ \bar{q}_2 \\ -8\bar{p}_2 \\ \bar{Q}-\bar{q}_1\bar{P} \\ -\frac{3}{4}\bar{q}_1^5-\frac{5}{4}\bar{q}_1\bar{q}_2^2-2\bar{c}-\bar{Q}^T\bar{P} \\ -\bar{p}_1-\frac{5}{4}\bar{q}_1^2\bar{q}_2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I_{N+2} & 0 \\ -I_{N+2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \nabla\left(\tfrac{1}{2}\bar{P}^T\Lambda\bar{P}-\tfrac{1}{2}\bar{Q}^T\bar{Q}+\bar{q}_1\bar{Q}^T\bar{P}\right.$$

$$\left.+\tfrac{3}{24}\bar{q}_1^6+\tfrac{5}{8}\bar{q}_1^2\bar{q}_2^2+\bar{q}_2\bar{p}_1-4\bar{p}_2^2+2\bar{c}\bar{q}_1\right) \equiv \bar{\Pi}_0\mathscr{H} \tag{4.14}$$

where $\bar{q}_1=v$, $\bar{q}_2=v_x$, $\bar{p}_1=-\frac{5}{4}v^2v_x+\frac{1}{8}v_{xxx}$, $\bar{p}_2=-\frac{1}{8}v_{xx}$, $\bar{c}=-\frac{1}{16}v_{4x}+\frac{5}{8}vv_x^2+\frac{5}{8}v^2v_{xx}-\frac{3}{8}v^5-\frac{1}{2}\bar{Q}^T\bar{P}$.

The Miura map linking both systems is

$$\begin{aligned} M:\ Q&=\bar{Q} \\ q_1&=-\bar{q}_1^2-\bar{q}_2 \\ q_2&=8\bar{p}_2-2\bar{q}_1\bar{q}_2 \\ P&=\bar{q}_1\bar{Q}+\Lambda\bar{P} \\ p_1&=\tfrac{1}{2}\bar{q}_1\bar{p}_1+\bar{q}_2\bar{p}_2-\tfrac{5}{16}\bar{q}_1\bar{q}_2^2-\tfrac{3}{16}\bar{q}_1^5-\tfrac{1}{2}\bar{c}-\tfrac{1}{4}\bar{Q}^T\bar{P} \\ p_2&=-\tfrac{1}{4}\bar{p}_1-\tfrac{5}{16}\bar{q}_1^2\bar{q}_2-\tfrac{1}{16}\bar{q}_2^2+\tfrac{1}{2}\bar{q}_1\bar{p}_2 \\ c&=\tfrac{1}{2}\bar{P}^T\Lambda\bar{P}-\tfrac{1}{2}\bar{Q}^T\bar{Q}+\bar{q}_1\bar{Q}^T\bar{P}+\tfrac{3}{24}\bar{q}_1^6+\tfrac{5}{8}\bar{q}_1^2\bar{q}_2^2+\bar{q}_2\bar{p}_1-4\bar{p}_2^2+2\bar{c}\bar{q}_1 \end{aligned} \tag{4.15}$$

and thus the second Hamiltonian structure of (4.13) is

$$\Pi_1 = M'\bar{\Pi}_0M'^T =$$

$$\left(\begin{array}{c|c|c|c|c|c|c} 0 & \multicolumn{2}{c|}{0} & \Lambda & -\frac{1}{4}Q & 0 & P \\ \hline 0 & 0 & -8 & 0 & q_1 & 0 & q_2 \\ & 8 & 0 & & -q_2 & -\frac{3}{2}q_1 & 32p_2 \\ \hline -\Lambda & \multicolumn{2}{c|}{0} & 0 & \frac{1}{4}P & -\frac{1}{4}Q & (\Lambda-q_1I_N)Q \\ \hline \frac{1}{4}Q^T & -q_1 & q_2 & -\frac{1}{4}P^T & 0 & \frac{1}{2}p_2+\frac{15}{64}q_1^2 & \frac{5}{16}q_1^3-\frac{5}{32}q_2^2-\frac{1}{2}c-\frac{1}{2}Q^TQ \\ \hline 0 & 0 & \frac{3}{2}q_1 & \frac{1}{4}Q^T & -\frac{1}{2}p_2-\frac{15}{64}q_1^2 & 0 & -\frac{5}{16}q_1q_2-p_1 \\ \hline -P^T & -q_2 & -32p_2 & -(\Lambda-q_1I_N)Q^T & -\frac{5}{16}q_1^3+\frac{5}{32}q_2^2+\frac{1}{2}c+\frac{1}{2}Q^TQ & \frac{15}{16}q_1q_2+p_1 & 0 \end{array}\right) \tag{4.16}$$

So, we have the following bi-Hamiltonian formulation of dynamics (4.13)

$$\mathscr{X}=\Pi_0\mathscr{H}_1=\Pi_1\mathscr{H}_0 \qquad \mathscr{H}_1=\mathscr{H},\ \mathscr{H}_0=c.$$

4.4. *The case of* $m=3$

Our last example is also the new one and concerns the restriction of (2.4) to

$$K_3[u]=\left(\tfrac{1}{64}u_{6x}+\tfrac{7}{32}uu_{4x}+\tfrac{7}{16}u_xu_{3x}+\tfrac{21}{64}u_{xx}^2+\tfrac{35}{32}u^2u_{xx}+\tfrac{35}{32}uu_x^2+\tfrac{35}{64}u^4\right)_x \tag{4.17}$$

Because of the length of the calculations, we quote here only the final results. The bi-Hamiltonian formulation of the dynamics (2.3, 2.4) with $m=3$ is the following

$$\begin{pmatrix} Q\\ q_1\\ q_2\\ q_3\\ P\\ p_1\\ p_2\\ p_3\\ c \end{pmatrix}_x = \begin{pmatrix} P\\ q_2\\ q_3\\ -128p_3\\ (\Lambda-q_1I_N)Q\\ -\frac{35}{64}q_1q_2^2+\frac{7}{128}q_3^2+\frac{35}{128}q_1^4-\frac{1}{2}c-\frac{1}{2}Q^TQ\\ -\frac{35}{64}q_1^2q_2-p_1\\ \frac{7}{64}q_1q_3-p_2\\ 0 \end{pmatrix}\equiv\mathscr{X}$$

$$\mathscr{X}=\Pi_0\mathscr{H}_1=\Pi_1\mathscr{H}_0 \tag{4.18}$$

where

$$\Pi_0=\begin{pmatrix} 0 & I_{N+3} & 0\\ -I_{N+3} & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}$$

$$\Pi_1=\left(\begin{array}{c|ccc|c|cccc} 0 & 0 & 0 & 0 & \Lambda & -\frac{1}{4}Q & 0 & 0 & P\\ \hline 0 & 0 & 0 & 0 & 0 & q_1 & 0 & \frac{1}{4} & q_2\\ 0 & 0 & 0 & 32 & 0 & \frac{3}{2}q_2 & q_1 & 0 & q_3\\ 0 & 0 & -32 & 0 & 0 & -\frac{3}{2}q_2 & \frac{5}{2}q_2 & -\frac{5}{2}q_1 & -128p_3\\ \hline -\Lambda & 0 & 0 & 0 & 0 & \frac{1}{4}P & -\frac{1}{4}Q & 0 & A\\ \hline \frac{1}{4}Q^T & -q_1 & -\frac{3}{2}q_2 & \frac{3}{2}q_3 & -\frac{1}{4}P^T & 0 & E & F & B\\ \hline 0 & 0 & -q_1 & -\frac{5}{2}q_2 & \frac{1}{4}Q^T & -E & 0 & G & C\\ 0 & -\frac{1}{4} & 0 & \frac{5}{2}q_1 & 0 & -F & -G & 0 & D\\ \hline -P^T & -q_2 & -q_3 & 128p_3 & -A & -B & -C & -D & 0 \end{array}\right)$$

$$A=(\Lambda-q_1I_N)Q$$

$$B=-\tfrac{35}{64}q_1q_2^2+\tfrac{7}{128}q_3^2+\tfrac{35}{128}q_1^4-\tfrac{1}{2}c$$

$$C = -\tfrac{35}{64}q_1^2q_2 - p_1$$

$$D = \tfrac{7}{64}q_1q_3 - p_2$$

$$E = \tfrac{1}{2}p_2 + \tfrac{35}{128}q_1^3 + \frac{35}{2\cdot 128}q_2^2$$

$$F = p_3 - \tfrac{35}{128}q_1q_2$$

$$G = \tfrac{1}{28}q_3 - \frac{35}{2\cdot 128}q_1^2$$

$$\mathscr{H}_0 = c$$

$$\mathscr{H}_1 = \tfrac{1}{2}P^TP - \tfrac{1}{2}Q^T\Lambda Q + \tfrac{1}{2}q_1Q^TQ - 64p_3^2 + q_2p_1 + q_3p_2 + \tfrac{35}{128}q_1^2q_2^2 - \tfrac{7}{128}q_1q_3^2 - \tfrac{7}{128}q_1^5 + \tfrac{1}{2}cq_1$$

$$q_1 = u \qquad q_2 = u_x \qquad q_3 = u_{xx} \qquad p_1 = -\tfrac{35}{64}u^2u_x - \tfrac{7}{64}u_xu_{xx} - \tfrac{7}{64}uu_{3x} - \tfrac{1}{128}u_{5x}$$

$$p_2 = \tfrac{7}{64}uu_{xx} + \tfrac{1}{128}u_{4x} \qquad p_3 = -\tfrac{1}{128}u_{3x}$$

$$c = \tfrac{1}{64}u_{6x} + \tfrac{7}{32}uu_{4x} + \tfrac{7}{16}u_xu_{3x} + \tfrac{21}{64}u_{xx}^2 + \tfrac{35}{32}u^2u_{xx} + \tfrac{35}{32}uu_x^2 + \tfrac{35}{64}u^4 - Q^TQ.$$

5. Conclusions

We will complete our paper with some comments on possible application of our method to other systems. Let

$$u_t = K_m[u] \qquad L(u)\psi = \lambda\psi \qquad u = (u_1, \ldots, u_n)^T \tag{5.1}$$

and

$$v_t = \bar{K}_m[v] \qquad \bar{L}(v)\varphi = \lambda\varphi \qquad v = (v_1, \ldots, v_n)^T \tag{5.2}$$

be two soliton hierarchies, related by some Miura map, and their Lax equations, related by respective gauge transformation. A general form of restricted flows for (5.1) and (5.2) reads

$$L(u)\psi_k = \lambda_k\psi_k \qquad k = 1, \ldots, N \qquad B_0\left(\sum_{k=1}^N \frac{\delta\lambda_k}{\delta u}\right) = K_m[u] \tag{5.3}$$

$$\bar{L}(v)\varphi_k = \lambda_k\varphi_k \qquad k = 1, \ldots, N \qquad \bar{B}_0\left(\sum_{k=1}^N \frac{\delta\lambda_k}{\delta v}\right) = \bar{K}_m[v] \tag{5.4}$$

where B_0, $\bar{B}_0$ are first Hamiltonian structures of (5.1) and (5.2), respectively.

The extension of our method, here applied to the KdV restricted flows, onto other restricted flows is possible if for some pair of infinite dimensional Hamiltonian soliton systems (5.1) and (5.2) related by a Miura map, the respective finite-dimensional restrictions (5.3) and (5.4) are also Hamiltonian and related by respective Miura map.

Following the procedure from section 2, let us first consider the uncoupled version of (5.3) and (5.4)

$$L(u \to 0)\psi_k = \lambda_k\psi_k \qquad K_m[u] = 0 \qquad k = 1, \ldots, N \tag{5.5}$$

$$\bar{L}(v \to 0)\varphi_k = \lambda_k\varphi_k \qquad \bar{K}_m[v] = 0 \qquad k = 1, \ldots, N. \tag{5.6}$$

The pair of asymptotic Lax equations is related by a map obtained from the asymptotic gauge transformation and in many cases forms a Hamiltonian system. The pair of second equations in (5.5)–(5.6) represents so-called stationary flows of systems (5.1)–(5.2). As was shown in [13, 14], the projection onto stationary flows preserves the Hamiltonian structure and Miura map for a wide class of soliton equations.

Now let us include the coupling between both systems. From the general form of the restricted flows (5.3)–(5.4) we find that the coupling terms are the same for the whole hierarchy, i.e. for arbitrary m. So, it is enough to prove that they are Hamiltonian and related by a Miura map for the simplest case $m=0$, i.e. when K_0 is a generatorof space translations (u_x in the KdV case). But this special case of restricted flows is known as a system obtained by the nonlinearization of Lax equation [3–5] (a Garnier system in the KdV case), and we know that for that special reduction the Hamiltonian structure and the Miura map are preserved in many cases as well (see for example [11]).

We can summarize our discussion by the statement that the reduction of systems (5.1)–(5.2) to their arbitrary restricted flows (5.3)–(5.4) preserves a Hamiltonian structure and a Miura map if both are preserved by a simplest restricted flow and by stationary flows, respectively. But as we know, this condition is fulfilled at least in some class of soliton systems, which makes our method more general. Possible candidates are restricted flows of dispersive water waves [9, 15, 16] and its modifications, for example.

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